30/10/14

Fitting the Pairing Force

Skyrme Interaction

$$V_{12}(r,r') = t_0 (1 + x_0 P^\sigma_{12}) \delta(r-r') + \frac{1}{2} t_1 (1 + x_1 P^\sigma_{12}) (\delta(r-r') k^2 + k'^2 \delta(r-r'))$$
$$+ t_2 (1 + x_2 P^\sigma_{12}) k' \cdot \delta(r-r') \cdot k + \frac{1}{6} t_3 \rho^\alpha (1 + x_3 P^\sigma_{12}) \delta(r-r')$$
$$+ i W_0 (\sigma_1 + \sigma_2)(k' \otimes \delta(r-r') k)$$

ONE-BODY WAVEFUNCTION : $\varphi_{m l j m_j}(\vec{r} \sigma q) = R_{m l j q}(r) \sum_{m_l} Y^l_{m_l}(\hat{r})\, C^{j m_j}_{l m_l \frac{1}{2} \sigma}$

Two-Body WAVEFUNCTION :

$$\Phi^{J M_J}_{m_1 l_1 j_1 m_{j_1}, m_2 l_2 j_2 m_{j_2}}(\vec{r}_1 \sigma_1 q_1, \vec{r}_2 \sigma_2 q_2) = R_{m_1 l_1 j_1 q_1}(r_1) R_{m_2 l_2 j_2 q_2}(r_2) \sum_{m_{j_1} m_{j_2}} C^{J M_J}_{j_1 m_{j_1} j_2 m_{j_2}} \sum_{m_{l1} m_{l2}} Y^{l_1}_{m_{l1}}(\hat{r}_1) Y^{l_2}_{m_{l2}}(\hat{r}_2)\, C^{j_1 m_{j_1}}_{l_1 m_{l1} \frac{1}{2} \sigma_1}\, C^{j_2 m_{j_2}}_{l_2 m_{l2} \frac{1}{2} \sigma_2}$$

A Cooper Pair is a pair coupled to $J = 0$

$$C^{00}_{j_1 m_{j_1}, j_2 m_{j_2}} = \frac{(-1)^{j - m_j}}{\sqrt{2j+1}} \delta_{j_1 j_2} \delta_{m_{j_1}, -m_{j_2}}$$

$$\Rightarrow \Phi^{00}_{m_1 l_1 j m_j m_2 l_2 j -m_j}(\vec{r}\sigma_1, \vec{r}\sigma_2) = R_{m_1 l_1 j q}(r_1) R_{m_2 l_2 j}(r_2) \frac{(-1)^{j - m_j}}{\sqrt{2j+1}} \sum_{m_{l1} m_{l2} m_j} Y^{l_1}_{m_{l1}}(\hat{r}_1) Y^{l_2}_{m_{l2}}(\hat{r}_2)\, C^{j m_j}_{l_1 m_{l1} \frac{1}{2} \sigma_1}\, C^{j -m_j}_{l_2 m_{l2} \frac{1}{2} \sigma_2}$$

and then antisymmetrized

$$\frac{1}{N^2} \int |\Phi^{00}_{\alpha}(r_1\sigma_1, r_2\sigma_2) - \Phi(r_2\sigma_2, r_1\sigma_1)|^2 = \frac{1}{N^2}\left(1 + 1 + 2\int \Phi^{00}_\alpha(1,2)\Phi^{00}_\alpha(2,1)\right) \equiv 1$$
$$\Rightarrow N^2 = 4 \Rightarrow N = 2$$

$$\langle r_1 r_2 | j^2; J=0 \rangle = \frac{1}{(\sqrt{2})^2}\left(\Phi^{00}_{m_1 l_1 j m_j m_2 l_2 j -m_j}(\vec{r}_1\sigma_1, \vec{r}_2\sigma_2) - \Phi^{00}_{m_1 l_1 j m_j m_2 l_2 j -m_j}(\vec{r}_2\sigma_2, \vec{r}_1\sigma_1)\right)$$
$$= \frac{(-1)^{j - m_j}}{\sqrt{2}\sqrt{2j+1}} \Big(R_{m_1 l_1 j}(r_1) R_{m_2 l_2 j}(r_2) \sum_{m_{l1} m_{l2} m_j} Y^{l_1}_{m_{l1}}(\hat{r}_1) Y^{l_2}_{m_{l2}}(\hat{r}_2)\, C^{j m_j}_{l_1 m_{l1} \frac{1}{2} \sigma_1} C^{j -m_j}_{l_2 m_{l2} \frac{1}{2} \sigma_2}$$
$$- R_{m_1 l_1 j}(r_2) R_{m_2 l_2 j}(r_1) \sum_{m_{l1} m_{l2}} Y^{l_1}_{m_{l1}}(\hat{r}_2) Y^{l_2}_{m_{l2}}(\hat{r}_1)\, C^{j m_j}_{l_1 m_{l1} \frac{1}{2} \sigma_2} C^{j -m_j}_{l_2 m_{l2} \frac{1}{2} \sigma_1} \Big)$$

δ interaction simplify the exchange of $\vec{r}_1, \vec{r}_2$

$$\langle j^2; J=0 | \delta(r - r') | j'^2; J=0 \rangle = \int r^2 R_{m_1 l_1 j}(r) R_{m_2 l_2 j}(r) R_{m_1' l_1' j'}(r) R_{m_2' l_2' j'}(r)\, dr$$
$$\int d\hat{r} \sum_{\substack{m_{l1} m_{l2} m_j \\ m_{l1}' m_{l2}' m_j'}} Y^{l_1 *}_{m_{l1}}(\hat{r}) Y^{l_2 *}_{m_{l2}}(\hat{r}) Y^{l'}_{m_{l1}'}(\hat{r}) Y^{l'}_{m_{l2}'}(\hat{r})\,.$$
$$\cdot \frac{(-1)^{2j - m_j - m_j'}}{4(2j+1)(\ast)} \left(C^{j m_j}_{l_1 m_{l1} \frac{1}{2} \sigma_1} C^{j -m_j}_{l_2 m_{l2} \frac{1}{2} \sigma_2} - C^{j m_j}_{l_1 m_{l1} \frac{1}{2} \sigma_2} C^{j -m_j}_{l_2 m_{l2} \frac{1}{2} \sigma_1} \right)$$
$$\left(C^{j' m_j'}_{l_1' m_{l1}' \frac{1}{2} \sigma_1'} C^{j' -m_j'}_{l_2' m_{l2}' \frac{1}{2} \sigma_2'} - C^{j' m_j'}_{l_1' m_{l1}' \frac{1}{2} \sigma_2'} C^{j' -m_j'}_{l_2' m_{l2}' \frac{1}{2} \sigma_1'} \right)$$

$= (\ast)$

$$Y^{\ell_1}_{m_{\ell_1}}(\hat{r})\, Y^{\ell_2}_{m_{\ell_2}}(\hat{r}) = \sum_{L\Pi} \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2L+1)}{4\pi}} \begin{pmatrix} \ell_1 & \ell_2 & L \\ m_{\ell_1} & m_{\ell_2} & \Pi \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_2 & L \\ 0 & 0 & 0 \end{pmatrix} Y^{L*(2)}_{\Pi}(\hat{r})$$

$$\Rightarrow \int d\hat{r}\, Y^{\ell_1 *}_{m_{\ell_1}}(\hat{r})\, Y^{\ell_2 *}_{m_{\ell_2}}(\hat{r})\, Y^{\ell_1'}_{m_{\ell_1}'}(\hat{r})\, Y^{\ell_2'}_{m_{\ell_2}'}(\hat{r}) = \sum_{\substack{LL' \\ \Pi\Pi'}} \frac{\sqrt{(2\ell_1+1)(2\ell_2+1)(2\ell_1'+1)(2\ell_2'+1)}}{4\pi}$$

$$\sqrt{(2L+1)(2L'+1)} \begin{pmatrix} \ell_1 & \ell_2 & L \\ m_{\ell_1} & m_{\ell_2} & \Pi \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_2 & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell_1' & \ell_2' & L' \\ m_{\ell_1}' & m_{\ell_2}' & \Pi' \end{pmatrix} \begin{pmatrix} \ell_1' & \ell_2' & L' \\ 0 & 0 & 0 \end{pmatrix} \cdot$$

$$\cdot \underbrace{\int Y^{(L)}_{\Pi}(\hat{r})\, Y^{(L')*}_{\Pi'}(\hat{r})\, d\hat{r}}_{\delta_{LL'}\,\delta_{\Pi\Pi'}}$$

$$= \sum_{\substack{L \\ \Pi}} \frac{2L+1}{4\pi} \sqrt{(2\ell_1+1)(2\ell_2+1)(2\ell_1'+1)(2\ell_2'+1)} \begin{pmatrix} \ell_1 & \ell_2 & L \\ m_{\ell_1} & m_{\ell_2} & \Pi \end{pmatrix} \begin{pmatrix} \ell_1' & \ell_2' & L' \\ m_{\ell_1}' & m_{\ell_2}' & \Pi' \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_2 & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell_1' & \ell_2' & L' \\ 0 & 0 & 0 \end{pmatrix}$$

$$C^{j m_j}_{\ell_1 m_{\ell_1} \frac{1}{2} \sigma_1} = (-1)^{+j+m_j} \begin{pmatrix} \ell_1 & \frac{1}{2} & j \\ m_{\ell_1} & \sigma_1 & -m_j \end{pmatrix} \sqrt{2j+1}$$

we have to consider
we are interested over a single j-shell $\Rightarrow \ell_1 = \ell_2 = \ell_1' = \ell_2'$

$$(*) = \langle j^2|_{J=0} \,|\, \delta \,|\, j^2|_{J=0} \rangle = (-1)^{+4j \cancel{- m_j + m_j} - m_j' + m_j'} \frac{(2j+1)}{4\,(4\pi)} (2\ell+1)^2 (-1)^{2j}$$

$$\int R^4_{n\ell j}(r) \;\ldots\; r^2\, dr$$

$$\cdot \sum_{L\Pi} \ldots (2L+1) \begin{pmatrix} \ell & \ell & L \\ 0 & 0 & 0 \end{pmatrix}^2 \cdot$$

$(*)$

$$\sum_{\substack{m_{\ell_1} m_{\ell_2} \\ m_j}} (-)^{m_j} \begin{pmatrix} \ell & \ell & L \\ m_{\ell_1} & m_{\ell_2} & \Pi \end{pmatrix} \left[\begin{pmatrix} \ell & \frac{1}{2} & j \\ m_{\ell_1} & \sigma_1 & -m_j \end{pmatrix} \begin{pmatrix} \ell & \frac{1}{2} & j \\ m_{\ell_2} & \sigma_2 & m_j \end{pmatrix} - \begin{pmatrix} \ell & \frac{1}{2} & j \\ m_{\ell_1} & \sigma_2 & -m_j \end{pmatrix} \begin{pmatrix} \ell & \frac{1}{2} & j \\ m_{\ell_2} & \sigma_1 & m_j \end{pmatrix} \right] \cdot$$

$$\sum_{\substack{m_{\ell_1}' m_{\ell_2}' \\ m_j'}} (-)^{m_j'} \begin{pmatrix} \ell & \ell & L \\ m_{\ell_1}' & m_{\ell_2}' & \Pi \end{pmatrix} \left[\begin{pmatrix} \ell & \frac{1}{2} & j \\ m_{\ell_1}' & \sigma_1' & -m_j' \end{pmatrix} \begin{pmatrix} \ell & \frac{1}{2} & j \\ m_{\ell_2}' & \sigma_2' & m_j' \end{pmatrix} - \begin{pmatrix} \ell & \frac{1}{2} & j \\ m_{\ell_1}' & \sigma_2' & -m_j' \end{pmatrix} \begin{pmatrix} \ell & \frac{1}{2} & j \\ m_{\ell_2}' & \sigma_1' & m_j' \end{pmatrix} \right]$$

$$(*)\ \sum_{m_{\ell_1} m_{\ell_2} m_j} (-)^{-m_j} \begin{pmatrix} \ell & \ell & L \\ m_{\ell_1} & m_{\ell_2} & \Pi \end{pmatrix} \begin{pmatrix} \ell & \frac{1}{2} & j \\ m_{\ell_1} & \sigma_1 & -m_j \end{pmatrix} \begin{pmatrix} \ell & \frac{1}{2} & j \\ m_{\ell_2} & \sigma_2 & m_j \end{pmatrix} =$$

$$= \sum_{m_{\ell_1} m_{\ell_2} m_j} (-)^{-m_j} \cancel{(-1)^{2\ell+L}} \begin{pmatrix} \ell & \ell & L \\ -m_{\ell_1} & -m_{\ell_2} & -\Pi \end{pmatrix} \begin{pmatrix} \ell & \frac{1}{2} & j \\ m_\ell & \sigma_1 & -m_j \end{pmatrix} (-1)^{(\ell+\frac{1}{2}+j)} \begin{pmatrix} \frac{1}{2} & \ell & j \\ \sigma_2 & m_{\ell_2} & m_j \end{pmatrix}$$

L is even

$\sigma_1 + m_{\ell_2} + \Pi$ = 0, 3j rules

$$= \sum_{m_{\ell_1} m_{\ell_2} m_j} (-1)^{\ell+\frac{1}{2}+j-m_{\ell_1}-m_{\ell_2}-m_j} \begin{pmatrix} \ell & \ell & L \\ -m_{\ell_1} & -m_{\ell_2} & -\Pi \end{pmatrix} \begin{pmatrix} \ell & \frac{1}{2} & j \\ m_\ell & \sigma_1 & -m_j \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \ell & j \\ \sigma_2 & m_{\ell_2} & m_j \end{pmatrix} \cdot (-1)^{-\Pi}$$

$$= (-1)^{-\Pi} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & L \\ \ell & \ell & j \end{Bmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & L \\ -\sigma_1 & -\sigma_2 & -\Pi \end{pmatrix} \to \frac{1}{2}-\frac{1}{2} \leq L \leq 2\cdot\frac{1}{2} \quad 0 \leq L \leq 1$$

but from $\begin{pmatrix} \ell & \ell & L \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow L$ must be even

$\Rightarrow L=0 \quad \Rightarrow \Pi = 0$

$$= \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \ell & \ell & j \end{Bmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\sigma_1 & -\sigma_2 & 0 \end{pmatrix} = - \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \ell & \ell & j \end{Bmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\sigma_2 & -\sigma_1 & 0 \end{pmatrix}$$

the exchange gives a $-$ for the antisymmetrization

$$\begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \ell & \ell & j \end{Bmatrix} \downarrow \frac{(-1)^{\frac{1}{2}+\ell+j}}{\sqrt{2\frac{1}{2}+1}\ \sqrt{2\ell+1}}$$

$\Rightarrow \sigma_1 = -\sigma_2$

$$\begin{pmatrix} s & s & 0 \\ \sigma & -\sigma & 0 \end{pmatrix} = \frac{(-1)^{s-\sigma}}{\sqrt{2s+1}}$$

$$= \frac{(-1)^{\frac{1}{2}+\ell+j+\frac{1}{2}-\sigma_1}}{2\sqrt{2\ell+1}}$$

$$(x) = \langle j^2; J=0 | \delta_r | j^2; J=0 \rangle = \frac{(2j+1)}{\cancel{4}(4\pi)} \int R^4_{n\ell j}(r)\, r^2 dr\ .$$

$$\cdot \cancel{(-1)^{2j}}\ \cancel{(2\ell+1)^2} \cdot \frac{1}{\cancel{2\ell+1}} \left(\frac{\cancel{(-1)^{1+2\ell+j}}}{\cancel{2\sqrt{2\ell+1}}}\ 2 \right)^2 \cdot \sum_{\sigma\sigma'} (-1)^{-\sigma} (-1)^{-\sigma'}$$

t_0 term in Skyrme

$$\Rightarrow \langle j^2; J=0 | \delta(r_{12})\, \delta_{\sigma\sigma'} | j^2; J=0 \rangle = (-\cancel{2}) \frac{(2j+1)}{\cancel{4}_2(4\pi)} \int R^4_{n\ell j}(r)\, r^2 dr$$

$t_0 x_0$ term in Skyrme

$$\langle j^2; J=0 | \delta(r_{12})\, P_r^\sigma | j^2; J=0 \rangle = (+\cancel{2}) \frac{(2j+1)}{\cancel{4}_2(4\pi)} \int R^4_{n\ell j}(r)\, r^2 dr$$