Gaussian Pairing Matrix Element Derivation

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1 Gaussian

The gaussian interaction between $\vec{r}$ and $\vec{r}'$ is given by,

$$g_a(\vec{r}-\vec{r}')=\frac{e^{|\vec{r}-\vec{r}'|^2/a^2}}{(a\sqrt{\pi})^3}=\frac{4}{a^3\sqrt{\pi}}e^{-\frac{r^2}{a^2}}e^{-\frac{r'^2}{a^2}}\sum_{LM}(-i)^L j_L\left(2i\frac{rr'}{a^2}\right)Y_M^{L^*}(\hat{r})Y_M^L(\hat{r'}) \tag{1}$$

where we have applied the multipole expansion [3] and j_L are the bessel functions

$$j_L(ix)=x^L\left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^L\frac{\sinh x}{x} \tag{2}$$

2 Two-body wavefunction

One body wavefunction is generally given by

$$\phi_{nljm_j}(\vec{r}\sigma)=\mathcal{R}_{nlj}(r)\sum_{m_l}\langle lm_l,1/2\sigma|jm_j\rangle Y_{m_l}^l(\hat{r}) \tag{3}$$

and two body Cooper pair wavefunction, so the coupled $J=0$, in a single shell

$$\Phi_{nljm_j}^{00}(\vec{r_1}\sigma_1,\vec{r_2}\sigma_2)=\frac{(-1)^{-j-m_j}}{\sqrt{2j+1}}\mathcal{R}_{nlj}(r_1)\mathcal{R}_{nlj}(r_1)\sum_{m_{l1}m_{l2}}\langle lm_{l1},1/2\sigma_1|jm_j\rangle\langle lm_{l2},1/2\sigma_2|j-m_j\rangle Y_{m_{l1}}^l(\hat{r}_1)Y_{m_{l2}}^l(\hat{r}_2) \tag{4}$$

that is already antisymmetrized since...

3 Two-body matrix element

3.1 Leading Order, no momenta terms

$$V_{LO}(\vec{r}_1,\vec{r}_2,\vec{r}_1',\vec{r}_2')=g_a(\vec{r}_1-\vec{r}_2)\delta(\vec{r}_1-\vec{r}_1')\delta(\vec{r}_2-\vec{r}_2')\left(\delta_{\sigma_1,\sigma_1'}\delta_{\sigma_2,\sigma_2'}+x_0P^\sigma\right) \tag{5}$$

where P^σ is the spin–exchange operator. We start keeping the case for general spin,

$$\begin{aligned}&\int \mathrm{d}^3r_1\mathrm{d}^3r_2\mathrm{d}^3r_1'\mathrm{d}^3r_2'\langle j^2;J=0|r_1r_2\rangle\langle r_1r_2|\tilde{V}_{LO}|r_1'r_2'\rangle\langle r_1'r_2'|j^2;J=0\rangle=\\ &=\int \mathrm{d}^3r_1\,\mathrm{d}^3r_2\frac{1}{4}\Phi_{nljm_j}^{00*}(\vec{r}_1\sigma_1,\vec{r}_2\sigma_2)g_a(\vec{r}_1-\vec{r}_2)\delta(\vec{r}_1-\vec{r}_1')\delta(\vec{r}_2-\vec{r}_2')\Phi_{nljm_j'}^{00}(\vec{r}_1\sigma_1',\vec{r}_2\sigma_2')\end{aligned} \tag{6}$$

$$\begin{aligned}&\int \mathrm{d}^3r_1\,\mathrm{d}^3r_2\Phi_{nljm_j}^{00*}(\vec{r}_1\sigma_1,\vec{r}_2\sigma_2)g_a(\vec{r}_1-\vec{r}_2)\Phi_{nljm_j'}^{00}(\vec{r}_1\sigma_1',\vec{r}_2\sigma_2')=\\
&=\int \mathrm{d}^3r_1\,\mathrm{d}^3r_2\frac{(-1)^{-2j-m_j-m_j'}}{2j+1}\mathcal{R}_{nlj}(r_1)\mathcal{R}_{nlj}(r_2)\sum_{m_{l1}m_{l2}}\langle lm_{l1},1/2\sigma_1|jm_j\rangle\langle lm_{l2},1/2\sigma_2|j-m_j\rangle Y_{m_{l1}}^{l*}(\hat{r}_1)Y_{m_{l2}}^{l*}(\hat{r}_2)\\
&\quad g_a(\vec{r}_1-\vec{r}_2)\mathcal{R}_{nlj}(r_1)\mathcal{R}_{nlj}(r_2)\sum_{m_{l1}'m_{l2}'}\langle lm_{l1}',1/2\sigma_1'|jm_j'\rangle\langle lm_{l2}',1/2\sigma_2'|j-m_j'\rangle Y_{m_{l1}'}^l(\hat{r}_1)Y_{m_{l2}'}^l(\hat{r}_2)\\
&=\int \mathrm{d}^3r_1\,\mathrm{d}^3r_2\frac{(-1)^{-2j-m_j-m_j'}}{2j+1}\mathcal{R}_{nlj}^2(r_1)\mathcal{R}_{nlj}^2(r_2)\sum_{m_{l1}m_{l2}}\langle lm_{l1},1/2\sigma_1|jm_j\rangle\langle lm_{l2},1/2\sigma_2|j-m_j\rangle Y_{m_{l1}}^{l*}(\hat{r}_1)Y_{m_{l2}}^{l*}(\hat{r}_2)\\
&\quad\frac{4}{a^3\sqrt{\pi}}e^{-\frac{r_1^2}{a^2}}e^{-\frac{r_2^2}{a^2}}\sum_{LM}(-i)^L j_L\left(2i\frac{r_1r_2}{a^2}\right)\frac{(-1)^{-2j-m_j-m_j'}}{2j+1}\sum_{m_{l1}'m_{l2}'}\langle lm_{l1}',1/2\sigma_1'|jm_j'\rangle\langle lm_{l2}',1/2\sigma_2'|j-m_j'\rangle Y_{m_{l1}'}^l(\hat{r}_1)Y_{m_{l2}'}^l(\hat{r}_2)\end{aligned}$$

$$=\frac{(-1)^{-2j-m_j-m_j'}}{2j+1}\frac{4}{a^3\sqrt{\pi}}\sum_{LM}(-i)^L \int dr_1\, dr_2 r_1^2 r_2^2 \mathcal{R}_{nlj}^2(r_1)\mathcal{R}_{nlj}^2(r_2) e^{-\frac{r_1^2}{a^2}} e^{-\frac{r_2^2}{a^2}} j_L\left(2i\frac{r_1 r_2}{a^2}\right)$$
$$\int d^2\hat{r}_1\, d\hat{r}_2 Y_M^{L^*}(\hat{r}_1) Y_M^L(\hat{r}_2) \sum_{m_{l1}m_{l2}} \langle l m_{l1}, 1/2\sigma_1 | j m_j\rangle \langle l m_{l2}, 1/2\sigma_2 | j-m_j\rangle Y_{m_{l1}}^{l*}(\hat{r}_1) Y_{m_{l2}}^{l*}(\hat{r}_2)$$
$$\sum_{m_{l1}'m_{l2}'} \langle l m_{l1}', 1/2\sigma_1' | j m_j'\rangle \langle l m_{l2}', 1/2\sigma_2' | j-m_j'\rangle Y_{m_{l1}'}^{l}(\hat{r}_1) Y_{m_{l2}'}^{l}(\hat{r}_2) \tag{7}$$

the angular part of Eq. (7) is given by

$$\sum_{LM,mm'} \langle l m_{l1}, 1/2\sigma_1 | j m_j\rangle \langle l m_{l2}, 1/2\sigma_2 | j-m_j\rangle \langle l m_{l1}', 1/2\sigma_1' | j m_j'\rangle \langle l m_{l2}', 1/2\sigma_2' | j-m_j'\rangle$$
$$\int d^2\hat{r}_1 Y_M^{L^*}(\hat{r}_1) Y_{m_{l1}}^{l*}(\hat{r}_1) Y_{m_{l1}'}^{l}(\hat{r}_1) \int d^2\hat{r}_2 Y_M^{L}(\hat{r}_2) Y_{m_{l2}}^{l*}(\hat{r}_2) Y_{m_{l2}'}^{l}(\hat{r}_2) \tag{8}$$

the integral of three spherical harmonics [1]

$$\int d^2\hat{r}_1 Y_M^{L*}(\hat{r}_1) Y_{m_{l1}}^{l*}(\hat{r}_1) Y_{m_{l1}'}^{l}(\hat{r}_1) = (-1)^{m_{l1}+M} \int d^2\hat{r}_1 Y_{-M}^{L}(\hat{r}_1) Y_{-m_{l1}}^{l}(\hat{r}_1) Y_{m_{l1}'}^{l}(\hat{r}_1) =$$
$$= (-1)^{m_{l1}+M}(2l+1)\sqrt{\frac{2L+1}{4\pi}} \begin{pmatrix} l & l & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l & L \\ -m_{l1} & m_{l1}' & -M \end{pmatrix}$$
$$\int d^2\hat{r}_1 Y_M^{L}(\hat{r}_1) Y_{m_{l2}}^{l*}(\hat{r}_1) Y_{m_{l2}'}^{l}(\hat{r}_1) = (-1)^{m_{l2}} \int d^2\hat{r}_1 Y_{M}^{L}(\hat{r}_1) Y_{-m_{l2}}^{l}(\hat{r}_1) Y_{m_{l2}'}^{l}(\hat{r}_1) =$$
$$= (-1)^{m_{l2}}(2l+1)\sqrt{\frac{2L+1}{4\pi}} \begin{pmatrix} l & l & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l & L \\ -m_{l2} & m_{l2}' & M \end{pmatrix} \tag{9}$$

thus Eq. (8) become

$$\sum_{LM,mm'} \langle l m_{l1}, 1/2\sigma_1 | j m_j\rangle \langle l m_{l2}, 1/2\sigma_2 | j-m_j\rangle \langle l m_{l1}', 1/2\sigma_1' | j m_j'\rangle \langle l m_{l2}', 1/2\sigma_2' | j-m_j'\rangle$$
$$(-1)^{m_{l1}+M}(2l+1)\sqrt{\frac{2L+1}{4\pi}} \begin{pmatrix} l & l & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l & L \\ -m_{l1} & m_{l1}' & -M \end{pmatrix} (-1)^{m_{l2}}(2l+1)\sqrt{\frac{2L+1}{4\pi}} \begin{pmatrix} l & l & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l & L \\ -m_{l2} & m_{l2}' & M \end{pmatrix} \tag{10}$$

using the relation between Clebsch-Gordan and Wigner 3j–symbols

$$\langle l m_{l1}, 1/2\sigma_1 | j m_j\rangle = (-1)^{j+m_j}\sqrt{2j+1} \begin{pmatrix} l & 1/2 & j \\ m_{l1} & \sigma_1 & -m_j \end{pmatrix} \tag{11}$$

that is difficult to solve for general case of spin σ_1, σ_1' so we start considering the term proportional to $\delta_{\sigma_1,\sigma_1'}\delta_{\sigma_2,\sigma_2'}$, thus $\sigma_1 = \sigma_1'$

$$\sum_{m_{l1},m_{l1}',\sigma_1} (-1)^{m_{l1}+M} \langle l m_{l1}, 1/2\sigma_1 | j m_j\rangle \langle l m_{l1}', 1/2\sigma_1 | j m_j'\rangle \begin{pmatrix} l & l & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l & L \\ m_{l1} & -m_{l1}' & M \end{pmatrix}$$
$$= (-1)^{2j}(2j+1) \begin{pmatrix} l & l & L \\ 0 & 0 & 0 \end{pmatrix} \sum_{m_{l1},m_{l2},\sigma_1} (-1)^{m_{l1}+M} \begin{pmatrix} l & 1/2 & j \\ m_{l1} & \sigma_1 & -m_j \end{pmatrix} \begin{pmatrix} l & 1/2 & j \\ m_{l1}' & \sigma_1 & -m_j' \end{pmatrix} \begin{pmatrix} l & l & L \\ m_{l1} & -m_{l1}' & M \end{pmatrix}$$
$$= -(-1)^{2j+M+m_j'-2l-1/2}(2j+1) \begin{pmatrix} l & l & L \\ 0 & 0 & 0 \end{pmatrix} \sum_{m_{l1},m_{l2},\sigma_1} (-1)^{-m_{l1}-m_{l1}'-\sigma_1+2l+1/2}$$
$$\begin{pmatrix} l & j & 1/2 \\ -m_{l1} & m_j & -\sigma_1 \end{pmatrix} \begin{pmatrix} l & 1/2 & j \\ m_{l1}' & \sigma_1 & -m_j' \end{pmatrix} \begin{pmatrix} l & l & L \\ m_{l1} & -m_{l1}' & M \end{pmatrix}$$
$$= -(-1)^{2j+M+m_j'-2l-1/2}(2j+1) \begin{pmatrix} l & l & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L & j & j \\ M & -m_j & m_j' \end{pmatrix} \begin{Bmatrix} L & j & j \\ 1/2 & l & l \end{Bmatrix} \tag{12}$$

And,

$$\sum_{m_{l2},m'_{l2},\sigma_2}(-1)^{m'_{l2}}\langle lm_{l2},1/2\sigma_2|j-m_j\rangle\langle lm'_{l2},1/2\sigma'_2|j-m'_j\rangle(-1)^{m_{l2}}-(2l+1)\sqrt{\frac{2L+1}{4\pi}}\begin{pmatrix} l & l & L \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} l & l & L \\ -m_{l2} & m'_{l2} & M \end{pmatrix}$$

$$=(-1)^{2j}(2j+1)\begin{pmatrix} l & l & L \\ 0 & 0 & 0 \end{pmatrix}\sum_{m_{l2},m'_{l2},\sigma_2}(-1)^{m_{l2}}\begin{pmatrix} l & 1/2 & j \\ m_{l2} & \sigma_2 & -m_j \end{pmatrix}\begin{pmatrix} l & 1/2 & j \\ m'_{l2} & \sigma_2 & -m'_j \end{pmatrix}\begin{pmatrix} l & l & L \\ m_{l2} & -m'_{l2} & -M \end{pmatrix}$$

$$=-(-1)^{2j+m'_j-2l-1/2}(2j+1)\begin{pmatrix} l & l & L \\ 0 & 0 & 0 \end{pmatrix}\sum_{m_{l2},m'_{l2},\sigma_1}-(-1)^{-m_{l2}-m'_{l2}-\sigma_2+2l+1/2}$$

$$\begin{pmatrix} l & j & 1/2 \\ -m_{l2} & m_j & -\sigma_1 \end{pmatrix}\begin{pmatrix} l & 1/2 & j \\ m'_{l2} & \sigma_1 & -m'_j \end{pmatrix}\begin{pmatrix} l & l & L \\ -m_{l2} & m'_{l2} & M \end{pmatrix}$$

$$=-(-1)^{2j+m'_j-2l-1/2}(2j+1)\begin{pmatrix} l & l & L \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} L & j & j \\ -M & m_j & -m'_j \end{pmatrix}\begin{Bmatrix} L & j & j \\ 1/2 & l & l \end{Bmatrix} \tag{13}$$

It follows that Eq. (8) becomes

$$=\sum_{m_j,m'_j}(-1)^{2j+m'_j-1/2}(2j+1)^2(2l+1)^2\frac{2L+1}{4\pi}\begin{pmatrix} l & l & L \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} L & j & j \\ M & -m_j & m'_j \end{pmatrix}\begin{Bmatrix} L & j & j \\ 1/2 & l & l \end{Bmatrix}$$

$$(-1)^{2j+M+m'_j-1/2}\begin{pmatrix} l & l & L \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} L & j & j \\ -M & m_j & -m'_j \end{pmatrix}\begin{Bmatrix} L & j & j \\ 1/2 & l & l \end{Bmatrix} \tag{14}$$

$2m'_j$ is odd, minus one is even. $4j$ and $2l$ are even, thus the phase is $=1$.

$$=(2j+1)^2(2l+1)^2\frac{2L+1}{4\pi}\begin{pmatrix} l & l & L \\ 0 & 0 & 0 \end{pmatrix}^2\begin{Bmatrix} L & j & j \\ 1/2 & l & l \end{Bmatrix}^2\sum_{m_j,m'_j}(2j+1)^2\begin{pmatrix} L & j & j \\ -M & m_j & -m'_j \end{pmatrix}\begin{pmatrix} L & j & j \\ M & -m_j & m'_j \end{pmatrix}=$$

$$=(2l+1)^2\frac{(-1)^{2j}}{4\pi}\begin{pmatrix} l & l & L \\ 0 & 0 & 0 \end{pmatrix}^2\begin{Bmatrix} L & j & j \\ 1/2 & l & l \end{Bmatrix}^2 \tag{15}$$

From [3] eq. (1.69) and 6j-properties

$$\begin{pmatrix} l & l & L \\ 0 & 0 & 0 \end{pmatrix}\begin{Bmatrix} L & j & j \\ 1/2 & l & l \end{Bmatrix}=\begin{pmatrix} l & l & L \\ 0 & 0 & 0 \end{pmatrix}\begin{Bmatrix} l & l & L \\ j & j & 1/2 \end{Bmatrix}=$$

$$=-\frac{1}{(2l+1)}\frac{1+(-1)^{2l+L}}{2}\begin{pmatrix} j & j & L \\ 1/2 & -1/2 & 0 \end{pmatrix}=-\frac{1}{(2l+1)}\begin{pmatrix} j & j & L \\ 1/2 & -1/2 & 0 \end{pmatrix} \tag{16}$$

with the prescription that L is even. Eq. (7) is then given by

$$\Rightarrow(7)=\sum_{L\text{ even},M}(-i)^L\frac{(-1)^{-2j}}{2j+1}\frac{4}{a^3\sqrt{\pi}}\int \mathrm{d}r_1\,\mathrm{d}r_2 r_1^2 r_2^2\mathcal{R}^2_{nlj}(r_1)\mathcal{R}^2_{nlj}(r_2)e^{-\frac{r_1^2}{a^2}}e^{-\frac{r_2^2}{a^2}}j_L\left(2i\frac{r_1r_2}{a^2}\right)$$

$$(2j+1)^2(2l+1)^2\frac{(-1)^{2j}}{4\pi}\frac{1}{(2l+1)^2}\begin{pmatrix} j & j & L \\ 1/2 & -1/2 & 0 \end{pmatrix}^2=$$

$$=\sum_{L\text{ even},M}(-i)^L(2j+1)\begin{pmatrix} j & j & L \\ 1/2 & -1/2 & 0 \end{pmatrix}^2\frac{1}{a^3\sqrt{\pi^3}}\int \mathrm{d}r_1\,\mathrm{d}r_2 r_1^2 r_2^2\mathcal{R}^2_{nlj}(r_1)\mathcal{R}^2_{nlj}(r_2)e^{-\frac{r_1^2}{a^2}}e^{-\frac{r_2^2}{a^2}}j_L\left(2i\frac{r_1r_2}{a^2}\right)=$$

$$=\frac{2j+1}{a^3\sqrt{\pi^3}}\sum_{L\text{ even}}(-i)^L(2L+1)\begin{pmatrix} j & j & L \\ 1/2 & -1/2 & 0 \end{pmatrix}^2\int \mathrm{d}r_1\,\mathrm{d}r_2 r_1^2 r_2^2\mathcal{R}^2_{nlj}(r_1)\mathcal{R}^2_{nlj}(r_2)e^{-\frac{r_1^2}{a^2}}e^{-\frac{r_2^2}{a^2}}j_L\left(2i\frac{r_1r_2}{a^2}\right) \tag{17}$$

References

[1] http://mathworld.wolfram.com/SphericalHarmonic.html

[2] http://www.phme.it/wilt/wp-content/uploads/sites/4/2014/11/Scanned-multifunction-device.pdf

[3] From nucleons to nucleus, J. Suhonen, Springer